1 3D Helmholtz Equation

A Green’s Function for the 3D Helmholtz equation must satisfy

\[ \nabla^2 G(r, r_0) + k^2 G(r, r_0) = \delta(r, r_0) \]

By Fourier transforming both sides of this equation, we can show that we may take the Green’s function to have the form

\[ G(r, r_0) = g(|r - r_0|) \]

and that

\[ g(r) = 4\pi \int_0^\infty \frac{\text{sinc}(2\pi \rho) \rho^2 d\rho}{k^2 - 4\pi^2 \rho^2} \]

First we take the Fourier transform of both sides:

\[ \mathcal{F}(\nabla^2 G(r, r_0) + k^2 G(r, r_0)) = e^{2\pi i r \rho} \]

(1)

(2\pi i \rho)^2 G(\rho, \rho_0) + k^2 G(\rho, \rho_0) = e^{2\pi i r \rho} \]

(2)

\[ G(\rho, \rho_0) \left[ (2\pi i \rho)^2 + k^2 \right] = e^{-2\pi i r \rho} \]

(3)

\[ G(\rho, \rho_0) = \frac{e^{-2\pi i r \rho}}{(2\pi i \rho)^2 + k^2} \]

(4)

\[ G(r, r_0) = \int_0^\pi \int_0^{2\pi} \int_0^\infty \frac{e^{-2\pi i r \rho} \rho^2 \sin \theta d\rho d\theta d\phi}{(2\pi i \rho)^2 + k^2} \]

(5)

\[ G(r, r_0) = 2\pi \int_0^\pi \int_0^\infty \frac{e^{-2\pi i r |r - r_0| \rho \cos \theta}}{k^2 - 4\pi^2 \rho^2} \rho^2 \sin \theta d\rho d\theta \]

(6)

\[ u = -\cos \theta \quad du = \sin \theta d\theta \]

(7)

\[ G(r, r_0) = 2\pi \int_{-1}^1 \int_0^\infty \frac{e^{-2\pi i r |r - r_0| \rho u}}{k^2 - 4\pi^2 \rho^2} \rho^2 d\rho du \]

(8)

\[ G(r, r_0) = 2\pi \int_0^\infty \frac{\rho^2}{k^2 - 4\pi^2 \rho^2} \int_{-1}^1 e^{-2\pi i r |r - r_0| \rho u} d\rho du \]

(9)

\[ G(r, r_0) = 2\pi \int_0^\infty \frac{\rho^2}{k^2 - 4\pi^2 \rho^2} \int_{-1}^1 \frac{1}{-2\pi i |r - r_0| \rho} \left( e^{-2\pi i |r - r_0| \rho} - e^{2\pi i |r - r_0| \rho} \right) d\rho \]

(10)

\[ G(r, r_0) = 2\pi \int_0^\infty \frac{\rho^2}{k^2 - 4\pi^2 \rho^2} \left[ \frac{2i \sin \left( -2\pi |r - r_0| \rho \right)}{-2\pi i |r - r_0| \rho} \right] d\rho \]

(11)
\[ G(r, r_0) = 2\pi \int_0^\infty \frac{\rho^2}{k^2 - 4\pi^2 \rho^2} \left[ \frac{\sin(-2\pi |r - r_0|\rho)}{-\pi |r - r_0|\rho} \right] d\rho \] (12)

\[ G(r, r_0) = 4\pi \int_0^\infty \frac{\rho^2}{k^2 - 4\pi^2 \rho^2} \sin(-2\pi |r - r_0|\rho) d\rho \] (13)

This means that little \( g \) has the expected form:

\[ g(r) = 4\pi \int_0^\infty \frac{\sin(2\pi \rho)}{k^2 - 4\pi^2 \rho^2} \rho^2 d\rho \] (14)

However, this integral passes through a singularity of the integrand. If we use the Cauchy principal value to deal with this problem, we use contour integration we find that the solution is proportional to a spherical wave from a monochromatic point source.

\[ \text{Re} \left\{ e^{ikr} \right\} \]

\[ g(r) = 4\pi \int_0^\infty \frac{\sin(2\pi \rho)}{k^2 - 4\pi^2 \rho^2} \rho^2 d\rho = 4\pi \int_0^\infty \frac{\sin(2\pi r\rho)}{(2\pi r\rho)(k^2 - 4\pi^2 \rho^2)} \rho^2 d\rho \] (15)

\[ g(r) = 2 \int_0^\infty \frac{\sin(2\pi r\rho)}{r(k^2 - 4\pi^2 \rho^2)} \rho d\rho \] (16)

We recognize that \( \text{sinc}(x) \) is an even function, so we can get the same result by integrating over infinite limits and halving the result.

\[ g(r) = \int_{-\infty}^\infty \frac{\sin(2\pi r\rho)}{r(k^2 - 4\pi^2 \rho^2)} \rho d\rho \] (17)

\[ g(r) = \text{Im} \left\{ \int_{-\infty}^\infty \frac{e^{2\pi i r\rho}}{r(k^2 - 4\pi^2 \rho^2)} \rho d\rho \right\} \] (18)

Now let us focus on solving the integral, then we will take the imaginary part at the end. We recognize two poles at \( \rho = \pm \frac{k}{2\pi} \). We must cleverly re-phrase the denominator in order to clearly cancel with our additional factor.

\[ \text{Res}_{k/2\pi} = \left. \left( \rho - \frac{k}{2\pi} \right) \frac{e^{2\pi i r\rho}}{-4\pi^2 r(\rho - \frac{k}{2\pi})(\rho + \frac{k}{2\pi})} \right|_{\rho=k/(2\pi)} \]

\[ = \left. \frac{e^{2\pi i r\rho}}{-4\pi^2 r(\rho + \frac{k}{2\pi})} \right|_{\rho=k/(2\pi)} \]

\[ = -4\pi^2 r(\frac{k}{2\pi} + \frac{k}{2\pi}) 2\pi \]

\[ = -4\pi rk \]

\[ = -8\pi^2 r \] (19)
\[
\text{Res}_{-k/2\pi} = \left( \rho + \frac{k}{2\pi} \right) e^{2\pi i r \rho} \bigg|_{\rho = -k/(2\pi)} - \frac{4\pi^2 r (\rho - \frac{k}{2\pi}) (\rho + \frac{k}{2\pi})}{\rho = -k/(2\pi)}
\]

\[
= -\frac{4\pi^2 r (\rho - \frac{k}{2\pi}) e^{-i k}}{\rho = -k/(2\pi)}
\]

\[
= -\frac{4\pi^2 r (\frac{k}{2\pi} - \frac{k}{2\pi})}{2\pi} e^{-i k}
\]

\[
= \frac{4\pi r k}{2\pi} e^{-i k}
\]

\[
= \frac{-k}{8\pi^2 r}
\]

Now we can utilize the residues, recognizing that since both lie on the line of integration, they have half the influence:

\[
\int_{-\infty}^{\infty} e^{2\pi i r \rho} r \frac{d\rho}{(k^2 - 4\pi^2 \rho^2)} = 2\pi i \sum_{i} \text{Residue}_i
\]

\[
2\pi i \sum_{i} \text{Residue}_i = 2\pi i \left( \frac{1}{2} e^{i k r} \frac{1}{-8\pi^2 r} + \frac{1}{2} e^{-i k r} \right) = i \left( \frac{e^{i k r}}{-8\pi r} + \frac{e^{-i k r}}{-8\pi r} \right)
\]

\[
2\pi i \sum_{i} \text{Residue}_i = \frac{i}{-8\pi r} (e^{i k r} + e^{-i k r}) = \frac{i}{-8\pi r} (\cos(kr) + i \sin(kr) + \cos(-kr) + i \sin(-kr))
\]

\[
2\pi i \sum_{i} \text{Residue}_i = \frac{i}{-8\pi r} (\cos(kr) + i \sin(kr) + \cos(kr) - i \sin(kr))
\]

\[
2\pi i \sum_{i} \text{Residue}_i = \frac{i}{-8\pi r} (\cos(kr) + \cos(kr)) = \frac{i \cos(kr)}{-4\pi r}
\]

Finally, we take the imaginary part in order to solve our original integral:

\[
g(r) = 4\pi \int_{0}^{\infty} \frac{\text{sinc}(2\rho \rho)}{\rho^2 - 4\pi^2 \rho^2} \frac{d\rho}{2\rho} = \frac{\cos(kr)}{-4\pi r}
\]

If we go ahead and plot the result of our operator acting upon this function, we do indeed find a delta function!

## 2 3D Wave Equation

A Green’s function for the 3D wave equation must satisfy

\[
\nabla^2 G(r, t, r_0, t_0) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} G(r, t, r_0, t_0) = \delta(r, r_0) \delta(t - t_0)
\]

We utilize the space-time Fourier transform, defined as:

\[
F(\rho, \omega) = \int_{\mathbb{R}^3} \int_{-\infty}^{\infty} f(r, t) e^{-2\pi i (r \rho - \omega t)} dtd^3r
\]
Using this Fourier transform we can show that we may assume the Greens’ function has the form:

\[ G(\mathbf{r}, t, \mathbf{r}_0, t_0) = g(|\mathbf{r} - \mathbf{r}_0|, t - t_0) \]

Instead of performing the whole Fourier transform at once, we instead perform just the time-dependent transform first.

\[ \mathcal{F}_t \left( \nabla^2 G(\mathbf{r}, t, \mathbf{r}_0, t_0) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} G(\mathbf{r}, t, \mathbf{r}_0, t_0) \right) = \delta(\mathbf{r}, \mathbf{r}_0)e^{2\pi it_0}\omega \]

If we make the substitution \( k = \frac{2\pi\omega}{c} \) we see that the wave equation is very similar to the Helmholtz equation we worked with in the previous section.

\[ \left( \nabla^2 + k^2 \right) G(\mathbf{r}, t, \mathbf{r}_0, t_0) = \delta(\mathbf{r}, \mathbf{r}_0)e^{2\pi it_0}\omega \]

Because of this similarity, we can utilize the solution all the way up to the point where we are inverse-transforming through the \( \omega \) variable in the next portion of the question. We can use contour integration to show that \( g(\mathbf{r}, t) \) is a linear combination of an incoming wave

\[ \frac{c}{r}\delta(r + ct) \]

and an outgoing wave

\[ \frac{c}{r}\delta(r - ct) \]

\[ g(\mathbf{r}) = 4\pi \int_{-\infty}^{\infty} \int_0^{\infty} \frac{\text{sinc}(2\pi \rho)}{k^2 - 4\pi^2 \rho^2} e^{2\pi it_0\omega} d\rho d\omega \]

\[ g(\mathbf{r}) = \int_{-\infty}^{\infty} -\cos(kr) e^{2\pi it_0\omega} d\omega \]

\[ g(\mathbf{r}) = \int_{-\infty}^{\infty} -\cos(kr) e^{2\pi it_0\omega} d\omega \]

\[ g(\mathbf{r}) = -\frac{1}{4\pi r} \int_{-\infty}^{\infty} \frac{1}{2} \left( e^{ikr} + e^{-ikr} \right) e^{2\pi it_0\omega} d\omega \]
\[ g(r) = -\frac{1}{8\pi r} \left( \int_{-\infty}^{\infty} e^{ikr} e^{2\pi i\omega t_0} d\omega + \int_{-\infty}^{\infty} e^{-ikr} e^{2\pi i\omega t_0} d\omega \right) \]  

(36)

\[ g(r) = -\frac{1}{8\pi r} \left( e^{\frac{2\pi i}{r}} \int_{-\infty}^{\infty} e^{2\pi i\omega t_0} d\omega + \int_{-\infty}^{\infty} e^{-\frac{2\pi i}{r}} e^{2\pi i\omega t_0} d\omega \right) \]  

(37)

\[ g(r) = -\frac{1}{8\pi r} \left( \int_{-\infty}^{\infty} e^{2\pi i\omega (t_0 + t)} d\omega + \int_{-\infty}^{\infty} e^{-2\pi i\omega (t_0 - t)} d\omega \right) \]  

(38)

\[ g(r) = -\frac{1}{4\pi r} \left[ \delta \left( \frac{r}{c} + t_0 \right) + \delta \left( \frac{r}{c} - t_0 \right) \right] \]  

(39)

\[ G(r) = -\frac{c}{4\pi r} \left[ \delta \left( |r - r_0| + c(t - t_0) \right) + \delta \left( |r - r_0| - c(t - t_0) \right) \right] \]  

(40)

To get this to resemble the form alluded to in the question, we must multiply by a form of one, and following the scaling rules for delta functions.

\[ G(r) = -\frac{c}{4\pi r} \left[ \delta \left( \frac{|r - r_0|}{c} + (t - t_0) \right) + \delta \left( \frac{|r - r_0|}{c} - (t - t_0) \right) \right] \]  

(41)

These are called incoming and outgoing waves because the “location” of the delta function either advances outward from the origin with time (corresponding to the \(r - ct\) argument) or shrinks in toward the origin as time grows (corresponding to the \(r + ct\) argument). Also a delta function of the radial coordinate looks like a “shell”, so the name fits quite well.